A NOTE ON FOURIER-JACOBI COEFFICIENTS OF SIEGEL MODULAR FORMS

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Abstract. In this note, we estimate the growth of the Petersson norms of Fourier-Jacobi coefficients f_m, for m's in arithmetic progressions, of Siegel cusp forms F of weight k and genus $n > 1$. As a consequence, we strengthen a result of Böcherer, Bruinier, and Kohnen in [1], and sharpen the result of Kohnen in [5].

1. Introduction

Let $\mathcal{H}_n$ be the Siegel upper half-plane of genus $n \geq 1$ and $\Gamma_n :=$ $\mathrm{Sp}_n(\mathbb{Z})$ be the full Siegel modular group. For $k, n \in \mathbb{N}$, let $S_k(\Gamma_n)$ denote the space of Siegel cusp forms of weight k on Γ_n.

If $Z \in \mathcal{H}_n$, then write $Z = \left(\begin{smallmatrix} \tau & z^t \\ z & \tau' \end{smallmatrix}\right)$, where $\tau \in \mathcal{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$, and $\tau' \in \mathcal{H}_1$. For any $F \in S_k(\Gamma_n)$ with $n > 1$, the Fourier-Jacobi expansion of $F(Z)$ relative to the maximal parabolic group of type $(n-1, 1)$ is of the form

$$F(Z) = \sum_{m \geq 1} f_m(\tau, z) e^{2\pi i m \tau'}.$$

The functions f_m belong to the space $J^{\mathrm{cusp}}_{k,m}$ of Jacobi cusp forms weight k, index m, and of genus $n-1$, i.e., invariant under the Jacobi group $\Gamma^J_{n-1} := \Gamma_{n-1} \ltimes \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$. For $f, g \in J^{\mathrm{cusp}}_{k,m}$, we let

$$\langle f, g \rangle = \int_{\Gamma^J_{n-1} \backslash \mathcal{H}^{n-1} \times \mathbb{C}^{n-1}} f(\tau, z) \overline{g(\tau, z)} (\det v)^{k-n-1} e^{-4\pi m v^{-1}[y^t]} du dv dx dy$$

be the inner product of f and g, and where $\tau = u + iv, z = x + iy$.

Let $q \geq 2$ be a natural number and $a \in \mathbb{Z}$ such that $(a, q) = 1$. In [1, Thm. 1], Böcherer, Bruinier, and Kohnen showed that for any non-zero function F in $S_k(\Gamma_n)(n > 1)$ with Fourier-Jacobi coefficients f_m, there exists infinitely many $m \in \mathbb{N}$ with $m \equiv a \pmod{q}$ such that $f_m \neq 0$, i.e., $\langle f_m, f_m \rangle \neq 0$. However, there was no information on the estimates of the Petersson norms of f_m, for $m \equiv a \pmod{q}$, from above or below

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with respect to m^{k-1}. For more details on estimating the Petersson norms of f_m with respect to $m^{k-\alpha}$, for $0 < \alpha < 1$, we refer the reader to the beautiful introduction of Kohnen in [5].

In this note, we strengthen this theorem by showing that there exists infinitely many $m \in \mathbb{N}$ with $m \equiv a \pmod q$ such that $\langle f_m, f_m \rangle > c_F m^{k-1}$ with $c_F > 0$ (see Theorem 3.1 in the text). In particular, this also sharpens the result in [5], where it was shown that there exists infinitely many $m \geq 1$ such that $\langle f_m, f_m \rangle > cm^{k-1}$, where $c > 0$ is essentially the Petersson norm of F, to m's in arithmetic progression.

Our results uses a melange of methods from [1], [5], and repeated application of the Main Lemma in [8].

2. Preliminaries

Let F be a non-zero cusp form in $S_k(\Gamma_n)(n > 1)$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. By Kohnen, Skoruppa [2] and Krieg [6], we know that $\langle f_m, f_m \rangle \ll_F m^k$ (the constant in $\ll$ depends only on F). Hence, for $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ such that $(a, q) = 1$, the Dirichlet series

$$D(s; a, q, F) := \sum_{\substack{m\geq 1 \\ m\equiv a \pmod q}} \frac{\langle f_m, f_m \rangle}{m^s}$$

converges for $s \in \mathbb{C}$ with $\Re(s) > k + 1$.

Proposition 2.1. *The Dirichlet series $D(s; a, q, F)$ converges for $\Re(s) > k$ and has only one real simple pole at $s = k$. Moreover, it vanishes at $s = 0, -1, -2, \cdots$.*

Proof. Let χ be a Dirichlet character mod q. The Dirichlet series

$$D(s, \chi, F) := \sum_{m\geq 1} \frac{\chi(m)\langle f_m, f_m \rangle}{m^s}$$

converges for $s \in \mathbb{C}$ with $\Re(s) \gg 0$. By [4],[5], the complete Dirichlet series

$$\mathbf{D}^*(s, \chi, F) := \left(\frac{2\pi}{q}\right)^{-2s} \Gamma(s)\Gamma(s-k+n)L(2s-2k+2n, \chi^2)\ D(s, \chi, F)$$

extends to a holomorphic function on $\mathbb{C}$ when $\chi \neq \chi_0$, where χ_0 is the principal Dirichlet character. On the other hand, when $\chi = \chi_0$, one knows that $\mathbf{D}^*(s, \chi_0, F)$ has a meromorphic continuation to $\mathbb{C}$ with a simple real pole at $s = k$ (see [4, page 495] and the remark in page 7 of [1]).

Recall that the poles of $\Gamma(s)$ are at $s = 0, -1, -2, \cdots$, and if $\chi^2 \neq \chi_0$, then the real zeros of $L(s, \chi^2)$ are at $s = 0, -2, -4, \cdots$, since χ^2 is an

even character. Moreover, all these zeros and poles are simple. With all these informations in hand, we can now study the nature of $D(s,\chi,F)$ and also about its real non-positive zeros.

If $\chi \neq \chi_0$, then $D(s,\chi,F)$ extends to a holomorphic function on $\mathbb{C}$ and vanishes at $s = 0, -1, -2, \cdots$.

If $\chi = \chi_0$, then $D(s,\chi_0,F)$ has a meromorphic continuation to $\mathbb{C}$ possibly with a simple real pole at $s = k$. Indeed, the function $D(s,\chi_0,F)$ has a simple real pole at $s = k$, since $\mathbf{D}^*(s,\chi_0,F)$ has a simple real pole at $s = k$ and none of the functions $L(2s-2k+2n,\chi^2), \Gamma(s-k+n), \Gamma(s)$ have a zero or pole at $s = k$ and they are holomorphic there. Furthermore, the series $D(s,\chi_0,F)$ vanishes at $s = 0, -1, -2, \cdots$.

Now, by the orthogonality of characters $\chi \pmod q$, we note that

$$\begin{aligned} D(s;a,q,F) &= \frac{1}{\varphi(q)} \sum_{m\geq 1} \sum_{\chi \pmod q} \chi(a^{-1}m)\langle f_m, f_m\rangle m^{-s} \\ &= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(a^{-1})\ D(s,\chi,F) \quad \text{for } \Re(s) > k. \end{aligned}$$

Hence, the Dirichlet series $D(s;a,q,F)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = k$ and vanishes at $s = 0, -1, -2, \cdots$. $\square$

We finish this section, by recalling a result on Dirichlet series with oscillating coefficients that was proved in [7], [8], which is an application of classical Landau's theorem.

Theorem 2.2 (Pribitkin [7],[8]). *Let $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a non-trivial Dirichlet series with $a_n \in \mathbb{R}$, and it converges somewhere. If $F(s)$ is holomorphic on the whole real line and has infinitely many real zeros, then the sequence $\{a_n\}_{n=1}^{\infty}$ is oscillatory, i.e., there exists infinitely many n such that $a_n > 0$, and similarly there exists infinitely many n such that $a_n < 0$.*

3. Statement and proof of the Main Result

Recall that, we are interested in estimating the growth of Petersson norms of Fourier-Jacobi coefficients f_m, of Siegel cusp forms F of weight k and genus $n > 1$, for m's in arithmetic progression. As a consequence, we strengthen Theorem 1 of [1].

By Proposition 2.1, we know that $D(s;a,q,F)$ has a pole at $s = k$, and let c_F denote the residue. Observe that c_F depends only on q, but we suppress this from the notation, as we dont need this fact. Recall that, $1 < q \in \mathbb{N}$ and $a \in \mathbb{Z}$ such that $(a,q) = 1$.

Theorem 3.1. *For $n > 1$, let F be a non-zero cusp form in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. Assume that c_F is real. Then c_F is positive. Moreover, there exists infinitely many m with $m \equiv a \pmod{q}$ for which $\langle f_m, f_m\rangle > c_F m^{k-1}$.*

Proof. Consider the Dirichlet series

$$\overline{D}(s; a, q, F) = D(s; a, q, F) - c_F\zeta(s - k + 1), \quad \Re(s) > k,$$

where $0 \neq c_F$ is the residue of $D(s; a, q, F)$ at $s = k$.

By Proposition 2.1, this series has a meromorphic continuation to $\mathbb{C}$ with no poles on the real line and vanishes at $s = k - 1 - 2t$, where $t \in \mathbb{N},\ t > (k-1)/2$.

For $m \geq 1$, let

$$\beta(m) \ := \ \begin{cases} \langle f_m, f_m\rangle - c_F m^{k-1} & \text{if } m \equiv a \pmod{q} \\ -c_F m^{k-1} & \text{otherwise} \end{cases}$$

be the general coefficient of $\overline{D}(s; a, q, F)$. The function $\bar{D}(s; a, q, F)$ cannot be identically trivial, since the $\beta(m)$'s are non-zero for $m \not\equiv a \pmod{q}$.

If $c_F < 0$, then $\beta(m) > 0$ for all m, which contradicts Theorem 2.2, which says that there are infinitely many sign changes for $\beta(m) (m \in \mathbb{N})$. Hence, the real number $c_F > 0$. In this case, again by Theorem 2.2, we see that there exists infinitely many m with $m \equiv a \pmod{q}$ such that $\langle f_m, f_m\rangle > c_F m^{k-1}$. □

In particular, the above theorem strengthens a theorem of Böcherer, Bruinier, and Kohnen on non-vanishing of Fourier-Jacobi coefficients of Siegel modular forms, see [1, Thm. 1].

We remark that a similar method, as above, may not be applicable in showing the existence of infinitely many m with $m \equiv a \pmod{q}$ such that $\langle f_m, f_m\rangle < c_F m^{k-1}$. However, if $q = 2$, then we can prove this statement. More precisely, we show:

Theorem 3.2. *For $n > 1$, let F be a non-zero cusp form in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. Let c'_F be the residue of the series $D(s; 1, 2, F)$ at $s = k$. Assume that c'_F is real. Then $c'_F > 0$, and there exists infinitely many $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ such that $\langle f_m, f_m\rangle > c'_F m^{k-1}$, and similarly for $\langle f_m, f_m\rangle < c'_F m^{k-1}$.*

Proof. Consider the Dirichlet series

$$\bar{D}(s; 1, 2, F) = D(s; 1, 2, F) - c'_F(1 - 2^{-(s-k+1)})\zeta(s - k + 1),$$

where $0 \neq c'_F$ is the residue of $D(s; 1, 2, F)$ at $s = k$.

For $m \geq 1$, let

$$\beta(m) \ := \ \begin{cases} \langle f_m, f_m \rangle - c'_F m^{k-1} & \text{if } m \equiv 1 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

be the general term of $\bar{D}(s; 1, 2, F)$.

By Proposition 2.1, this series has a meromorphic continuation to $\mathbb{C}$ with no poles on the real line and vanishes at $s = k - 1 - 2t$, where $t \in \mathbb{N},\ t > (k-1)/2$.

If $c'_F < 0$, then $\beta(m) > 0$ for all $m \geq 1$, which contradicts Theorem 2.2. Hence the residue $c'_F > 0$. Now, we show that the function $\bar{D}(s; 1, 2, F)$ cannot be identically trivial. Unlike in the proof of Theorem 3.1, here we need a different argument to prove this statement. If the function $\bar{D}(s; 1, 2, F)$ identically trivial, then

$$D(s; 1, 2, F) = c'_F (1 - 2^{-(s-k+1)}) \zeta(s - k + 1).$$

Now, evaluate the above expression at $s = 0$ (resp., at $s = -1$) if k is even (resp., if k is odd), to see that $c'_F = 0$, since the series $D(s; 1, 2, F)$ has zeros at $s = 0, -1$, by Proposition 2.1. Again by the same proposition, we see that c'_F cannot be equal to zero, as $D(s; 1, 2, F)$ has a pole at $s = k$.

Since $\bar{D}(s; 1, 2, F)$ is non-trivial, the theorem follows from Theorem 2.2, which states that there are infinitely many sign changes for $\beta(m) (m \geq 1)$. Hence, there exists infinitely many $m \equiv 1 \pmod 2$ such that $\langle f_m, f_m \rangle > c'_F m^{k-1}$, and similarly for $\langle f_m, f_m \rangle < c'_F m^{k-1}$. □

In the proof above, we have used that $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$. This proof can't be generalized for $q \neq 2$, since $\zeta(s, a)/\zeta(s)$ is entire if and only if $a = 1/2$ or 1, where $\zeta(s, a)$ is the Hurwitz zeta-function (cf. [9]).

4. When the residue $c_F \in \mathbb{C} \setminus \mathbb{R}$:

In this section, we consider the case when the residue c_F of $D(s; a, q, F)$ at $s = k$ belongs to $\mathbb{C} \setminus \mathbb{R}$. In this case also, we can obtain results which are similar to the ones in the previous sections.

Theorem 4.1. *For $n > 1$, let F be a non-zero cusp form in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients $\{f_m\}_{m \in \mathbb{N}}$. Let c_F be the residue of the Dirichlet series $D(s; a, q, F)$ at $s = k$. If $c_F \in \mathbb{C} \setminus \mathbb{R}$, then $\Re(c_F) > 0$, and there exists infinitely many m with $m \equiv a \pmod q$ for which $\langle f_m, f_m \rangle > \Re(c_F) m^{k-1}$.*

Proof. Suppose that $c_F = c + iy$ with $c, y \in \mathbb{R}, y \neq 0$. Consider the Dirichlet series

$$\overline{D}(s; a, q, F) = D(s; a, q, F) - c_F \zeta(s - k + 1) \ \text{ for } \Re(s) > k.$$

First note that $c \neq 0$. If not, by Theorem 2.2, we see that $\langle f_m, f_m \rangle = 0$ for all $m \equiv a \pmod{q}$. This is a contradiction to [1, Lem. 3].

By arguing as in Theorem 3.1, one can show that c cannot be negative. Hence, $c = \Re(c_F) > 0$. Now the theorem follows by the Main Lemma in [8]. Hence, there are infinitely many m with $m \equiv a \pmod{q}$ such that $\langle f_m, f_m \rangle > \Re(c_F) m^{k-1}$. □

We finish this note with the following theorem, which is a generalization of Theorem 3.2. The proof of this theorem is similar to the proofs of Theorem 3.2 and Theorem 4.1, hence we omit the proof.

Theorem 4.2. *Let F be a non-zero cusp form in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients f_m $(m \geq 1)$. Let q be a prime number and let c_F be the residue of $D(s; a, q, F)$ for some $1 \leq a \leq q-1$ (hence for all a). Then $\Re(c_F) > 0$ and there exists $1 \leq b, c \leq q-1$ such that the following holds:*

- *there exists infinitely many $m \in \mathbb{N}$ with $m \equiv b \pmod{q}$ such that $\langle f_m, f_m \rangle > \Re(c_F) m^{k-1}$, and*
- *there exists infinitely many $m \in \mathbb{N}$ with $m \equiv c \pmod{q}$ such that $\langle f_m, f_m \rangle < \Re(c_F) m^{k-1}$.*

Corollary 4.3. *If $q = 2$ and $c_F \in \mathbb{R}$, then Theorem 3.2 is a special case of the theorem above.*

References

[1] Böcherer, S.; Bruinier, J. H.; Kohnen, W. Non-vanishing of scalar products of Fourier-Jacobi coefficients of Siegel cusp forms. Math. Ann. 313 (1999), no. 1, 1–13.

[2] Kohnen, W.; Skoruppa, N.-P. A certain Dirichlet series attached to Siegel modular forms of degree two. Invent. Math. 95 (1989), no. 3, 541–558.

[3] Kohnen, Winfried. On characteristic twists of certain Dirichlet series. Mem. Fac. Sci. Kyushu Univ. Ser. A 47 (1993), no. 1, 103–117.

[4] Kohnen, W.; Krieg, A.; Sengupta, J. Characteristic twists of a Dirichlet series for Siegel cusp forms. Manuscripta Math. 87 (1995), no. 4, 489–499.

[5] Kohnen, Winfried. On the growth of the Petersson norms of Fourier-Jacobi coefficients of Siegel cusp forms. Bull. Lond. Math. Soc. 43 (2011), no. 4, 717–720.

[6] Krieg, Aloys. A Dirichlet series for modular forms of degree n. Acta Arith. 59 (1991), no. 3, 243–259

[7] Pribitkin, Wladimir de Azevedo. On the sign changes of coefficients of general Dirichlet series. Proc. Amer. Math. Soc. 136 (2008), no. 9, 3089–3094.

[8] Pribitkin, Wladimir de Azevedo. On the oscillatory behavior of certain arithmetic functions associated with automorphic forms. J. Number Theory 131 (2011), no. 11, 2047–2060.

[9] Steuding, J. On the value-distribution of the Hurwitz zeta-function at the non-trivial zeros of the Riemann zeta-function. Abhdlg. Math. Sem. Uni. Hamburg 71 (2001), 113-121.

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